# Nonlinear Averaging Applied to the Control of Pulse Width Modulated (PWM) Pneumatic Systems

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***Abstract*****—**This paper presents a control methodology that enables nonlinear model based sliding mode control of PWM-controlled pneumatic servo actuators. An averaging approach is developed to describe the equivalent continuous-time dynamics of a PWM controlled nonlinear system, which renders the originally discontinuous system, possibly non-affine in the input, into an equivalent system that is in control canonical nonlinear form. A sliding mode controller is then developed for a pneumatic servo actuator based on the averaged equivalent control canonical model. The controller is implemented on an experimental system, and the effectiveness of the proposed approach validated by experimental tracking data.

## I. Introduction

The servo control of pneumatic actuators is typically implemented by utilizing some type of servovalve to control the airflow into and out of the respective sides of a pneumatic cylinder. Several researchers have studied the design and control of such systems, including Shearer [1-3], Mannetje [4], Wang et al. [5], Bobrow and McDonell [6], and Richer and Hurmuzlu [7-8], whose studies involved servocontrol via spool-type four-way servovalves, and Jacobsen et al. [9], Ben-Dov and Salcudean [10], and Henri et al. [11], whose studies involved control via flapper or jet-pipe type servovalves. In such systems, the cost of the servovalve in nearly all cases dominates the cost of the actuator.

Pulse width modulated (PWM) control offers the ability to provide servocontrol of pneumatic actuators at a significantly lower cost by utilizing binary solenoid valves in place of costly servovalves. As is the case with control via servovalves, several researchers have investigated the use of PWM control of solenoid on/off valves for the servo control of pneumatic actuators. In particular, Noritsugu [12-13], Kunt and Singh [14], Ye et al. [15], and Shih and Hwang [16] incorporate principally heuristic approaches for the PWM control of pneumatic servo systems. Such approaches do not offer the stability or performance guarantees of approaches developed within a more rigorous analytical framework. The work of van Varseveld and Bone [17] proposes the use of discrete time linear controllers for the PWM control of pneumatic servo systems. Barth et al. [18] utilize a linear state-space averaging technique to enable the design of a linear compensator via a loop shaping approach that provides a prescribed degree of stability robustness in addition to a desired closed-loop bandwidth. None of these prior works, however, treats the PWM control pneumatic servo systems in the context of model-based nonlinear control. Due to their highly nonlinear nature, pneumatic servo systems are particular well suited to the use of nonlinear model-based controllers, such as sliding mode control. Paul et al. [19] proposed a switching controller (not technically a PWM controller) based on a "reduced-order" nonlinear model that provides stability in the sense of Lyapunov. The "reduced-order" aspect of their approach, however, requires simplifying assumptions, which cannot accommodate the full nonlinear character of a pneumatic servo system. In particular, they neglect the nonlinearity in the pressure dynamics and the distinction between the choked and unchoked flow regimes through the solenoid valves, which are two of the most significant nonlinearities in such systems.

Unlike these prior works, this paper presents a method for nonlinear model based PWM control of a pneumatic servo actuator based on the full nonlinear model of such systems. Specifically, this paper extends the authors' previously published averaging techniques [18] to nonlinear systems. The nonlinear averaging technique is then utilized as the basis for the development of a PWM-based sliding mode approach to the control of pneumatic servo systems, which is based on the full (i.e., non-reduced-order) nonlinear description of such systems. The controller is implemented on a single degree-of-freedom pneumatic servo system, and the effectiveness of the method verified by experimental trajectory tracking data.

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## II. Average Model Based PWM-Control of Nonlinear Systems

A pulse width modulation (PWM) control system meters the power delivered to an actuator from a power source in discrete packets, as opposed to the continuous delivery of power characteristic in a continuous control system (i.e., those treated in [1-11]). The "packets" of power delivered by a PWM system, however, are in essence averaged by the dynamics of the system being controlled. As such, the resulting dynamics, on the characteristic time-scales of interest for the closed-loop system, can be described by the average dynamics of such systems. The authors previously utilized an averaging approach to derive, for a PWM controlled linear system dynamics, the equivalent continuous-time dynamics of the linear system, and implemented this approach for the control of a linearized pneumatic servo actuator [18]. Such an approach is extended here to nonlinear system dynamics, so that PWM-based controllers can be developed within the context of nonlinear model based control techniques. The averaged continuous time dynamics for a PWM controlled nonlinear system dynamics is developed as follows.

Consider a general nonlinear dynamic system, non-affine in its m control variables, exhibiting controllable switching, and given in regular form as,

$$x^{(n)} = f_i(\mathbf{x},\mathbf{u}) \tag{1}$$

where $x^{(n)}$ is the nth derivative of $x$, the vector $\mathbf{x}$ contains the continuous states of the system, including all lower derivatives of $x$, the control vector is $\mathbf{u} = [u_1\ u_2 \cdots u_m]$, and the form of the function $f_i$ is dependent on certain ranges of the control variables. Without loss of generality, consider a switching system with two possible input vectors denoted as $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, each referred to as an input mode. The dynamics of the system operating in each mode can be written as:

$$\text{Mode 1:}\quad x^{(n)} = f_1(\mathbf{x},\mathbf{u}_1) \tag{2}$$

$$\text{Mode 2:}\quad x^{(n)} = f_2(\mathbf{x},\mathbf{u}_2) \tag{3}$$

A state averaged model of this system operating with a duty cycle d can be written,

$$x^{(n)} = f_1 d + f_2(1-d) \tag{4}$$

where $\mathbf{u}_1$ and therefore Mode 1 is the effective fraction $d$ of a fixed period T, and $\mathbf{u}_2$ and therefore Mode 2 is the effective fraction $(1-d)$ of the PWM period T. To avail this state-space average model to standard nonlinear control methodologies, (4) can be recast into the following regular form, affine in the new control variable $d$,

$$x^{(n)} = \tilde{f}(\mathbf{x},\mathbf{u}_2) + \tilde{b}(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2)d \tag{5}$$

where

$$\tilde{f}(\mathbf{x},\mathbf{u}_2) = f_2(\mathbf{x},\mathbf{u}_2)$$

$$\tilde{b}(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2) = f_1(\mathbf{x},\mathbf{u}_1) - f_2(\mathbf{x},\mathbf{u}_2)$$

and the input is confined to a saturated range $d \in [0, 1]$. Equation (5) not only serves to transform a non-continuous control problem into a continuous one, but also serves to convert a general nonlinear system, controllably switching in nature, and non-affine in its (potentially) multiple control variables, into a continuous single input nonlinear system that is affine in the new control variable d specifying the duty cycle of the PWM signal. Thus, an original discrete-input, possibly multi-input and possibly non-affine nonlinear system has been converted to an equivalent continuous-time, single-input nonlinear system in a canonical nonlinear control form which is well-suited to many nonlinear control approaches (e.g., sliding mode control, integrator back-stepping control). The following two sections develop a nonlinear model and sliding mode controller for a pneumatic servo system, based on the averaging approach described by (5).

## III. A Nonlinear Averaged Model of a PWM-Controlled Pneumatic Servo System

As previously described by (5), one can convert discontinuous nonlinear dynamics from discrete inputs to a continuous output into an equivalent continuous dynamics from duty cycle input to continuous output. In the case of a pneumatic servo system controlled by solenoid valves, the discrete inputs are the valve spool positions, and the continuous output is the motion of the load. In order to obtain the continuous description from the duty cycle to output, the behavior of the system in each discrete mode must first be described. This section of the paper derives a model of the pneumatic servo system, defines and describes the discrete modes of operation of this system, and finally derives the equivalent continuous description from duty cycle to motion based on the averaging methods previously described.

Assuming air is a perfect gas undergoing an isothermal process, the rate of change of the pressure inside each chamber of the cylinder can be expressed as:

$$\dot{P}_{(a,b)} = \frac{RT}{V_{(a,b)}}\left(\dot{m}_{in(a,b)} - \dot{m}_{out(a,b)}\right) - \frac{P_{(a,b)}}{V_{(a,b)}}\dot{V}_{(a,b)} \tag{6}$$

where $P_{(a,b)}$ is the absolute pressure inside each side of the cylinder, $\dot{m}_{in(a,b)}$ and $\dot{m}_{out(a,b)}$ are the mass flow rates into and out of each side of the cylinder, $R$ is the universal gas constant, $T$ is the fluid temperature, and $V_{(a,b)}$ is the volume of each cylinder chamber. Based on isentropic flow assumptions, the mass flow rate though a valve orifice with effective area Av for a compressible substance will reside in either a sonic (choked) or subsonic (unchoked) flow regime:

$$\dot{m}(P_u,P_d)=\begin{cases}\sqrt{\frac{k}{RT}\left(\frac{2}{k+1}\right)^{(k+1)/(k-1)}}C_f P_u A_v \\ \quad \text{if } \frac{P_d}{P_u}\le C_r \text{ (choked)} \\ \sqrt{\frac{2k}{RT(k-1)}}\sqrt{1-\left(\frac{P_d}{P_u}\right)^{(k-1)/k}}\left(\frac{P_d}{P_u}\right)^{(1/k)} C_f P_u A_v \\ \quad \text{otherwise (unchoked)}\end{cases} \tag{7}$$

where $C_f$ is the discharge coefficient of the valve, $P_u$ and $P_d$ are the upstream and downstream pressures, respectively, $k$ is the ratio of specific heats, $C_r$ is the pressure ratio that divides the flow regimes into unchoked and choked flow. Assuming a combined inertial and viscous load, the motion dynamics of the system shown in Fig. 1 can be written as:

$$M\ddot{x}+B\dot{x}=P_aA_a-P_bA_b-P_{atm}A_r \tag{8}$$

where $M$ is the payload plus the piston and rod assembly mass, $B$ is the viscous friction coefficient, $A_a$ and $A_b$ are the effective areas of each side of the piston, and $A_r$ is the cross-sectional area of the piston rod. The total dynamics from the valve area as input (which is an algebraic function of the spool position) to the motion output is given by the combination of (6-8):

$$\dddot{x}=\frac{A_a}{M}\left[\frac{RT}{V_a}\left(\dot{m}_{in,a}-\dot{m}_{out,a}\right)-\frac{P_a}{V_a}\dot{V}_a\right] -\frac{A_b}{M}\left[\frac{RT}{V_b}\left(\dot{m}_{in,b}-\dot{m}_{out,b}\right)-\frac{P_b}{V_b}\dot{V}_b\right]-\frac{B}{M}\ddot{x} \tag{9}$$

where $\dot{m}_{in,a}$, $\dot{m}_{out,a}$, $\dot{m}_{in,b}$, $\dot{m}_{out,b}$ are the mass flow rates into and out of the two sides of the cylinder, which are functions of the valve areas as expressed by (7). As such, the input vector is defined as:

$$\mathbf{u}=\begin{bmatrix}A_{v,in,a} & A_{v,out,a} & A_{v,in,b} & A_{v,out,b}\end{bmatrix}^{\mathrm{T}} \tag{10}$$

where $A_{v,in,a}$ and $A_{v,in,b}$ are the valve areas between the pressure supply and chambers a and b, respectively, and $A_{v,out,a}$ and $A_{v,out,b}$ are the valve areas between the respective chambers and atmosphere. The pneumatic servo system under consideration in this paper incorporates two two-position three-way solenoid valves, such that at any given time, each chamber can either be connected to supply or exhaust (atmosphere). As such, the system has four possible modes as follows:

$$\text{Mode 1: } \mathbf{u}_1=A_v\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^{\mathrm{T}} \tag{11}$$

$$\text{Mode 2: } \mathbf{u}_2=A_v\begin{bmatrix}0 & 1 & 0 & 1\end{bmatrix}^{\mathrm{T}} \tag{12}$$

$$\text{Mode 3: } \mathbf{u}_3=A_v\begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^{\mathrm{T}} \tag{13}$$

$$\text{Mode 4: } \mathbf{u}_4=A_v\begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}^{\mathrm{T}} \tag{14}$$

where Mode 1 corresponds to charging side a and discharging b, Mode 2 corresponds to discharging both, Mode 3 corresponds to charging side b and discharging a, Mode 4 corresponds to charging both, and $A_v$ is the valve area of the open valves. Since, in the presence of Mode 2, Mode 4 does not offer significant utility with respect to tracking control, Mode 4 is not considered further. Thus, the controller can assume any of the first three modes. Further, within a single PWM period, the controller will assume either a combination of Modes 1 and 2, which corresponds to a control effort in one direction, or a combination of Modes 2 and 3, which corresponds to a control effort in the opposite direction. As such, denoting $d_i$ as the fraction of time in Mode $i$ within a control cycle, the controller will assume one of the following two combinations:

$$d_1+d_2=1 \text{ and } d_3=0 \tag{15}$$

or

$$d_2+d_3=1 \text{ and } d_1=0 \tag{16}$$

Thus, the PWM duty cycle d can be defined as:

$$d=\begin{cases}d_1 & \text{if } d_1\neq 0\\ -d_3 & \text{if } d_3\neq 0\end{cases} \tag{17}$$

The corresponding average model can be developed based on (1-5) as:

$$\dddot{x}=\begin{cases}f(x)+b^+(x)d & \text{if } d\ge 0\\ f(x)+b^-(x)d & \text{if } d<0\end{cases} \tag{18}$$

where

$$f(x)=-\frac{A_a}{M}\left[\frac{RT}{V_a}\dot{m}(P_a,P_{atm})+\frac{P_a}{V_a}\dot{V}_a\right] +\frac{A_b}{M}\left[\frac{RT}{V_b}\dot{m}(P_b,P_{atm})+\frac{P_b}{V_b}\dot{V}_b\right]-\frac{B}{M}\ddot{x} \tag{19}$$

$$b^+(x)=\frac{A_a}{M}\frac{RT}{V_a}[\dot{m}(P_s,P_a)+\dot{m}(P_a,P_{atm})] \tag{20}$$

$$b^-(x)=\frac{A_b}{M}\frac{RT}{V_b}[\dot{m}(P_s,P_b)+\dot{m}(P_b,P_{atm})] \tag{21}$$

## IV. Pulse-Width-Modulated Sliding Mode Control

Having expressed the PWM system dynamics in a continuous input canonical form, a sliding mode control approach can be applied to the control of the system. Selecting an integral sliding surface as:

$$s=\left(\frac{d}{dt}+\lambda\right)^3\int_0^t e\,d\tau \tag{22}$$

where $e=x-x_{desired}$ and $\lambda$ is a control gain, a robust control law can be developed based on a standard sliding mode approach, which results in a robust control law in terms of the duty cycle as follows:

$$d^{+/-}=\frac{\dddot{x}_d-f(x)-3\lambda\ddot{e}-3\lambda^2\dot{e}-\lambda^3 e}{b^{+/-}(x)}-K^{+/-}\,\mathrm{sgn}(s) \tag{23}$$

where, for the controller proposed herein, the robustness gain K is time variant according to

$$K^{+/-}=\frac{\beta^{+/-}(F+\eta)+(\beta^{+/-}-1)(x_d-f(x)-3\lambda\ddot{e}-3\lambda^2\dot{e}-\lambda^3 e)}{b^{+/-}(x)} \tag{24}$$

where $\beta^{+/-}=(\left|b^{+/-}\right|^{max}/\left|b^{+/-}\right|^{min})^{1/2}$, and $F=0.1|f|$.

It should be noted that though two possible values $d^+$ and $d^-$ can be calculated from (23), only the value corresponding to the correct sign assumption is used. Additionally, the duty cycle must be saturated at unity (i.e., 100% is the greatest possible duty cycle). As such, the duty cycle is selected as:

$$d=\begin{cases} sat(d^+) & if\ d^+\geq -d^- \\ -sat(-d^-) & if\ d^+<-d^- \end{cases} \tag{25}$$

where sat(x) is the saturated value between 0 and 1.

## V. EXPERIMENTAL RESULTS

The proposed control approach was implemented on an experimental setup in order to validate the proposed approach. The setup, which is shown schematically in Fig. 1, incorporates a 2.7 cm (1-1/16 in) inner diameter, 10 cm (4-in) stroke pneumatic cylinder (Numatics 1062D04-04A) is commanded to drive a 10kg moving mass. The control valves are two pilot-assisted 3-way solenoid-activated valves (SMC VQ1200H-5B) operating at a PWM frequency of 25 Hz. The system is powered by an air supply at the absolute pressure of 584 kPa (85 psi).

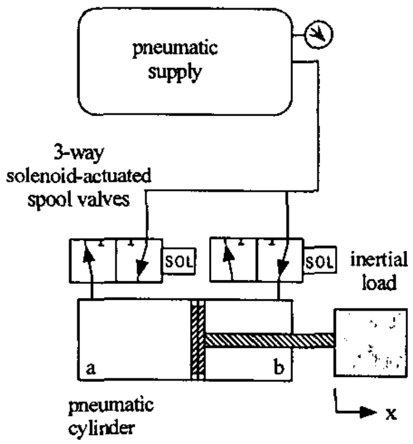


Fig. 1. Solenoid valve controlled pneumatic servoactuator driving an inertial load

Fig. 2, 3 and 4 show the measured responses to sinusoid inputs at three different frequencies. As shown, the controller approach provides effective motion tracking via solenoid on/off valves. Tracking performance was degraded at higher frequencies, presumably due to some combination of choked flow through the valves (which limits the actuation power) and their limited switching response time. Regarding the latter, as with any PWM controlled system, the closed-loop system bandwidth is limited to approximately an order of magnitude below the PWM switching frequency, which in this case was 25 Hz, limited by the dynamic limitations of the valves. Thus, even without the mass flow saturation (i.e., choked flow), it is unlikely that this system could track frequencies much greater than 2 Hz.

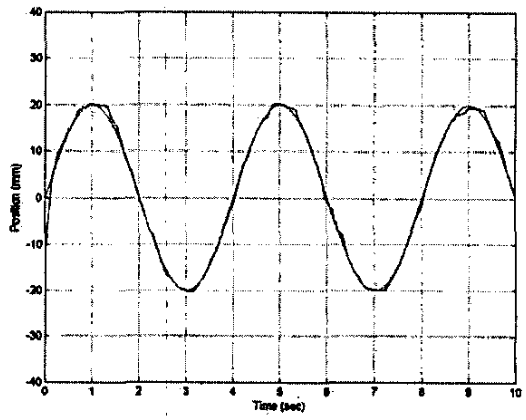


Fig. 2. Tracking performance at 0.25 Hz

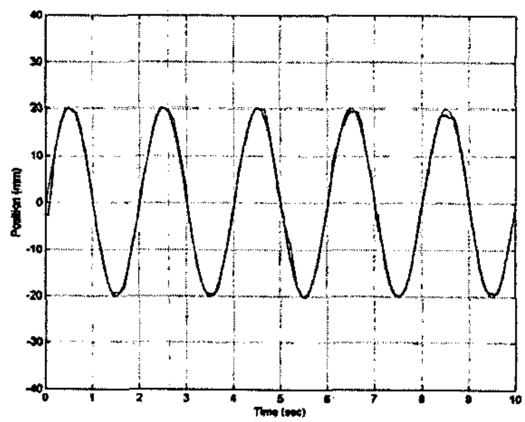


Fig. 3. Tracking performance at 0.5 Hz

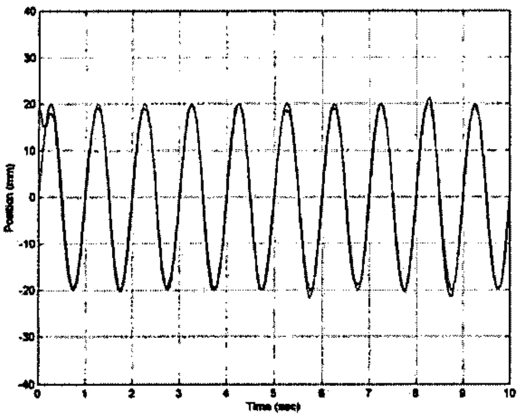


Fig. 4. Tracking performance at 1 Hz

## VI. Conclusion

This paper presented a control approach capable of obtaining robust servo control via relatively inexpensive on/off solenoid valves. A nonlinear model averaging approach is developed that enables the use of a full (non-reduced-order) nonlinear model based control. This averaging method is applied to a PWM controlled pneumatic servo system, which in turn enables the application of a sliding mode controller. The proposed controller is implemented on an experimental setup and shown to provide effective control.

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